

Robust recovery-type *a posteriori* error estimators for streamline upwind/Petrov Galerkin discretizations for singularly perturbed problems

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Abstract

In this paper, we investigate adaptive streamline upwind/Petrov Galerkin (SUPG) methods for singularly perturbed convection-diffusion-reaction equations in a new dual norm presented in [15]. The flux is recovered by either local averaging in conforming $H(\text{div})$ spaces or weighted global L^2 projection onto conforming $H(\text{div})$ spaces. We further introduce a recovery stabilization procedure, and develop completely robust *a posteriori* error estimators with respect to the singular perturbation parameter ε . Numerical experiments are reported to support the theoretical results and to show that the estimated errors depend on the degrees of freedom uniformly in ε .

Keywords: singular perturbation, streamline upwind/Petrov Galerkin method, recovery-type *a posteriori* error estimator, robust.

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1 Introduction

Let Ω be a bounded polygonal or polyhedral domain in \mathbb{R}^d ($d = 2$ or 3) with Lipschitz boundary $\Gamma = \Gamma_D \cup \Gamma_N$, where $\Gamma_D \cap \Gamma_N = \emptyset$. Consider the following stationary singularly perturbed convection-diffusion-reaction problem

$$\begin{cases} \mathcal{L}u := -\varepsilon \Delta u + \mathbf{a} \cdot \nabla u + bu = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \varepsilon \frac{\partial u}{\partial \mathbf{n}} = g & \text{on } \Gamma_N, \end{cases} \quad (1.1)$$

where $0 < \varepsilon \ll 1$ is the singular perturbation parameter, $\mathbf{a} \in (W^{1,\infty}(\Omega))^d$, $b \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, \mathbf{n} is the outward unit normal vector to Γ , and equation (1.1) is scaled such that $\|\mathbf{a}\|_{L^\infty} = \mathcal{O}(1)$ and $\|b\|_{L^\infty} = \mathcal{O}(1)$. The Dirichlet boundary Γ_D has a positive $(d-1)$ -dimensional Lebesgue measure, which includes the inflow boundary $\{\mathbf{x} \in \partial\Omega : \mathbf{a}(\mathbf{x}) \cdot \mathbf{n} < 0\}$. Assume that there are two nonnegative constants β and c_b , independent of ε , satisfying

$$b - \frac{1}{2} \nabla \cdot \mathbf{a} \geq \beta \quad \text{and} \quad \|b\|_{L^\infty(\Omega)} \leq c_b \beta. \quad (1.2)$$

Note that if $\beta = 0$, then $b \equiv 0$ and there is no reaction term in (1.1).

Adaptive finite element methods (FEMs) for numerical solutions of partial differential equations (PDEs) are very popular in scientific and engineering computations. *A posteriori* error estimation is an essential ingredient of adaptivity. Error estimators in literature can be categorized into three classes: residual based, gradient recovery based, and hierarchical bases based. Each approach has certain advantages.

Designing a robust *a posteriori* error estimator for singularly perturbed equations is challenging, because the estimators usually depend on the small diffusion parameter ε . This problem was first investigated by Verfürth [27], in which both upper and lower bounds for error estimator in an ε -weighted energy norm was proposed. It was shown that the estimator was robust when the local Péclet number is not very large. Generalization of this approach can be found

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in, e.g., [8, 19, 22, 24]. He considered also robust estimators in an ad hoc norm in [28]. In [25], Sangalli pointed out that the ad hoc norm may not be appropriate for problem (1.1), and proposed a residual-type *a posteriori* estimator for 1D convection-diffusion problem which is robust up to a logarithmic factor with respect to global Péclet number. Recently, John and Novo [18] proposed a robust *a posteriori* error estimator in the natural SUPG norm (used in the *a priori* analysis) under some hypotheses, which, however, may not be fulfilled in practise. In [2], a fully computable, guaranteed upper bounds are developed for the discretisation error in energy norm. Very recently, Tobiska and Verfürth [26] presented robust residual *a priori* error estimates for a wide range of stabilized FEMs.

For *a posteriori* error estimation of singularly perturbed problems, it is crucial to employ an appropriate norm, since the efficiency of a robust estimator depends fully on the norm. Du and Zhang [15] proposed a dual norm, which is induced by an ε -weighted energy norm and a related $H^{1/2}(\Omega)$ -norm. A uniformly robust *a posteriori* estimator for the numerical error was obtained from the new norm. Both theoretical and numerical results showed that the estimator performs better than the existing ones in the literature.

It is well known that the *a posteriori* error estimators of the recovery type possess many appealing properties, including simplicity, universality, and asymptotical exactness, which lead to their widespread adoption, especially in the engineering community (cf., e.g., [3, 4, 7, 11, 30, 31, 32, 33]). However, when applied to many problems of practical interest, such as interface singularities, discontinuities in the form of shock-like fronts, and of interior or boundary layers, they lose not only asymptotical exactness but also efficiency on relatively coarse meshes. They may overrefine regions where there are no error, and hence fail to reduce the global error (see [6, 20, 21]). To overcome this difficulty, Cai and Zhang [9] developed a global recovery approach for the interface problem. The flux is recovered in $H(\text{div})$ conforming finite element (FE) spaces, such as the Raviart-Thomas (RT) or the Brezzi-Douglas-Marini (BDM) spaces, by global weighted L^2 -projection or local averaging. The resulting recovery-based (implicit and explicit) estimators are measured in the standard energy norm, which turned out to be robust if the diffusion coefficient is monotonically distributed.

This approach was further extended for solving general second-order elliptic PDEs [10]. The implicit estimators based on the L^2 -projection and $H(\text{div})$ recovery procedures were proposed to be the sum of the error in the standard energy norm and the error of the recovered flux in a weighted $H(\text{div})$ norm. The global reliability and the local efficiency bounds for these estimators were established. For singularly perturbed problems, the estimators developed in [9, 10] are not robust with respect to ε . To the authors' knowledge, no robust recovery-type estimators have been proposed for such problems in the literature.

Motivated by aforementioned works, we extend the approach in [15] and develop robust recovery-based *a posteriori* error estimators for the SUPG method for singularly perturbed problems. Three procedures will be applied, which are the explicit recovery through local averaging in RT_0 spaces, the implicit recovery based on the global weighted L^2 -projection in RT_0 and BDM_1 spaces, and the implicit $H(\text{div})$ recovery procedure. Numerical errors will be measured in a dual norm presented in [15]. Note that these estimators are different from those in [15], since the jump in the normal component of the flux consists of a recovery indicator in addition to an incidental term (see Remark 4.1). Our recovery procedures are also different from those in [9, 10] (e.g., the flux recovery based on the local averaging provides an appropriate choice of weight factor, the $H(\text{div})$ recovery procedure develops a stabilization technique, the recovery procedures treat Neumann boundary conditions properly, etc.). Moreover, the estimators developed here are uniformly robust with respect to ε and β .

The rest of this paper is organized as follows. In Section 2, we introduce the variational formulation and some preliminary results. In Section 3, we define an implicit flux recovery procedure based on the L^2 -projection onto the lowest-order RT or BDM spaces, and an explicit recovery procedure through local averaging in the lowest-order RT spaces. In Section 4, for implicit and explicit recovery procedures, we give a reliable upper bound for the numerical error in a dual norm developed in [15]. Section 5 is devoted to the analysis of efficiency of the estimators. Here, the efficiency is in the sense that the converse estimate of upper bound holds up to different higher order terms (usually oscillations of data) and a different multiplicative constant depends only on the shape of the mesh. We show that the estimators are completely robust with respect to ε and β . In Section 6, we define a stabilization $H(\text{div})$ recover procedure, and develop robust recovery-based estimator by using the main results of Sections 4 and 5. Numerical tests are provided in Section 7 to support the theoretical results.

2 Variational Formulation and Preliminary Results

For any subdomain ω of Ω with a Lipschitz boundary γ , denote by $\langle \cdot, \cdot \rangle_e$ and $(\cdot, \cdot)_\omega$ the inner products on $e \subseteq \gamma$ and ω , respectively. Throughout this paper, standard notations for Lebesgue and Sobolev spaces and their norms and seminorms are used [1]. In particular, for $1 \leq p < \infty$ and $0 < s < 1$, the norm of the fractional Sobolev space $W^{s,p}(\omega)$

is defined as

$$||v||_{W^{s,p}(\omega)} := \left\{ ||v||_{L^p(\omega)}^p + \int_{\omega} \int_{\omega} \frac{|v(x) - v(y)|^p}{|x - y|^{d+ps}} dx dy \right\}^{1/p} \text{ for } v \in W^{s,p}(\omega).$$

When $p = 2$, we write $H^s(\omega)$ for $W^{s,2}(\omega)$. We will also use the space $H(\text{div}; \omega) := \{\tau \in L^2(\omega)^d : \nabla \cdot \tau \in L^2(\omega)\}$. To simplify notations, we write $||\cdot||_{s,\omega} = ||\cdot||_{H^s(\omega)}$, $|\cdot|_{s,\omega} = |\cdot|_{H^s(\omega)}$, and $||\cdot||_{\gamma} = ||\cdot||_{L^2(\gamma)}$. Moreover, when no confusion may arise, we will omit the subindex Ω in the norm and inner product notations if $\omega = \Omega$. Let $H_D^1(\Omega) := \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$. Define a bilinear form $B(\cdot, \cdot)$ on $H_D^1(\Omega) \times H_D^1(\Omega)$ by

$$B(u, v) = \varepsilon(\nabla u, \nabla v) + (\mathbf{a} \cdot \nabla u, v) + (bu, v). \quad (2.1)$$

The variational formulation of (1.1) is to find $u \in H_D^1(\Omega)$ such that

$$B(u, v) = (f, v) + \langle g, v \rangle_{\Gamma_N} \quad \forall v \in H_D^1(\Omega). \quad (2.2)$$

Under the assumption (1.2), equation (2.2) possesses a unique weak solution (cf., e.g., [23]).

Let \mathcal{T}_h be a shape regular admissible triangulation of Ω into triangles or tetrahedra satisfying the angle condition [12]. We use $F \preceq G$ to represent $F \leq CG$, and write $F \approx G$ if both $F \preceq G$ and $G \preceq F$ hold true. Here and in what follows, we use C for a generic positive constant depending only on element shape regularity and d . Assume that \mathcal{T}_h aligns with the partition of Γ_D and Γ_N . Let \mathcal{E} be the set of all *edges* (for $d = 2$) or *faces* (for $d = 3$) of elements in \mathcal{T}_h . Then $\mathcal{E} = \mathcal{E}_{\Omega} \cup \mathcal{E}_D \cup \mathcal{E}_N$, where \mathcal{E}_{Ω} is the set of interior edges/faces, and \mathcal{E}_D and \mathcal{E}_N are the sets of boundary edges/faces on Γ_D and Γ_N , respectively. Let $P_k(K)$ be the space of polynomials on K of total degree at most k . Let the FE space V_h be

$$V_h := \{v_h \in C(\overline{\Omega}) : v_h|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h, v_h|_{\Gamma_D} = 0\}.$$

Define a bilinear form $B_{\delta}(\cdot, \cdot)$ on $V_h \times V_h$ and a linear functional $l_{\delta}(\cdot)$ on V_h by

$$\begin{aligned} B_{\delta}(u_h, v_h) &= B(u_h, v_h) + \sum_{K \in \mathcal{T}_h} \delta_K (-\varepsilon \Delta u_h + \mathbf{a} \cdot \nabla u_h + bu_h, \mathbf{a} \cdot \nabla v_h)_K, \\ l_{\delta}(v_h) &= (f, v_h) + \langle g, v_h \rangle_{\Gamma_N} + \sum_{K \in \mathcal{T}_h} \delta_K (f, \mathbf{a} \cdot \nabla v_h)_K, \end{aligned}$$

where δ_K 's are nonnegative stabilization parameters satisfying

$$\delta_K ||\mathbf{a}||_{L^{\infty}(K)} \leq Ch_K \quad \forall K \in \mathcal{T}_h. \quad (2.3)$$

Note that Δu_h is interpreted as the Laplacian applied to $u_h|_K, \forall K \in \mathcal{T}_h$. For the lowest-order element, though Δu_h vanishes on each element, we will keep this term for complete presentation of the SUPG method and its analysis in below (cf. Section 5).

Then the FE approximation of (1.1) is to find $u_h \in V_h$ such that

$$B_{\delta}(u_h, v_h) = l_{\delta}(v_h) \quad \forall v_h \in V_h. \quad (2.4)$$

Note that the choice $\delta_K = 0$ for all $K \in \mathcal{T}_h$ yields the standard Galerkin method, and the choice $\delta_K > 0$ for all K corresponds to the SUPG-discretization. The existence and uniqueness of solution to (2.4) are guaranteed by (1.2) and (2.3) (cf., e.g., [16, 17, 28]).

Define an ε -weighted energy norm by

$$||v||_{\varepsilon} := (\varepsilon |v|_1^2 + \beta ||v||^2)^{1/2} \quad \forall v \in H^1(\Omega).$$

Let $h(\mathbf{x})$ be a function satisfying $0 < h_{\min} \leq h(\mathbf{x}) \leq h_{\max} < \infty$ almost everywhere in Ω . Define a norm of $v \in H_D^1(\Omega)$ with respect to $h(\mathbf{x})$ by

$$\begin{aligned} |||v|||^2 &:= ||v||_{\varepsilon}^2 + \max \{ ||v||_{H^{1/2}(\Omega)}^2, ||h(\mathbf{x})^{-1/2} v||^2 + ||h(\mathbf{x})^{1/2} \nabla v||^2 \} \\ \text{or} \quad |||v|||^2 &:= ||v||_{\varepsilon}^2 + ||h(\mathbf{x})^{-1/2} v||^2 + ||h(\mathbf{x})^{1/2} \nabla v||^2. \end{aligned}$$

It is shown [15] that the dual norm

$$||| \cdot |||_* := \sup_{v \in H_D^1(\Omega) \setminus \{0\}} \frac{B(\cdot, v)}{|||v|||} \quad (2.5)$$

induced by the bilinear form (2.1) satisfies, for $u \in H_D^1(\Omega)$,

$$|||u|||_* = \sup_{v \in H_D^1(\Omega) \setminus \{0\}} \frac{\langle \mathcal{L}u, v \rangle}{|||v|||} \geq \frac{|||u|||}{\|\mathcal{L}^{-1}\|_{(H_D^1(\Omega))^* \rightarrow H_D^1(\Omega)}}.$$

This inequality shows that $|||u|||_*$ may reflect the first derivatives of u even if $\varepsilon = 0$.

Let $I_h : L^2(\Omega) \rightarrow V_h$ be the Cl  ment interpolation operator (cf. [13, 28, 25] and [12, Exercise 3.2.3]). The following estimates on I_h are found in [15].

Lemma 2.1. *Let h_e be the diameter of an edge/face e . For any $v \in H_D^1(\Omega)$,*

$$\sum_{K \in \mathcal{T}_h} \delta_K^2 \max\{\beta, \varepsilon h_K^{-2}, h_K^{-1}\} \|\mathbf{a} \cdot \nabla(I_h v)\|_K^2 \preceq |||v|||^2, \quad (2.6)$$

$$\sum_{K \in \mathcal{T}_h} \max\{\beta, \varepsilon h_K^{-2}, h_K^{-1}\} \|v - I_h v\|_K^2 \preceq |||v|||^2, \quad (2.7)$$

$$\sum_{e \in \Gamma_N} \max\{\varepsilon^{1/2} \beta^{1/2}, \varepsilon h_e^{-1}, 1\} \|v - I_h v\|_e^2 \preceq |||v|||^2. \quad (2.8)$$

Remark 2.2 (On the norm $||| \cdot |||_*$). *We first review a robust residual-based a posteriori estimator, which is proposed in SUPG norm under some hypotheses [18]. Let*

$$\eta_1 = \left(\sum_{K \in \mathcal{T}_h} \min \left\{ \frac{C}{\beta}, C \frac{h_K^2}{\varepsilon}, 24\delta_K \right\} \|R_K\|_K^2 \right)^{1/2}, \quad \eta_2 = \left(\sum_{K \in \mathcal{T}_h} 24\delta_K \|R_K\|_K^2 \right)^{1/2},$$

$$\text{and } \eta_3 = \left(\sum_{e \in \mathcal{E}} \min \left\{ \frac{24}{\|\mathbf{a}\|_{\infty, e}}, C \frac{h_e}{\varepsilon}, \frac{C}{\varepsilon^{1/2} \beta^{1/2}} \right\} \|R_e\|_e^2 \right)^{1/2},$$

where the cell residual R_K and edge/face residual R_e are defined by (4.1) and

$$R_e := \begin{cases} -[\varepsilon \nabla u_h \cdot \mathbf{n}_e]|_e & \text{if } e \not\subset \Gamma, \\ g - \varepsilon \nabla u_h \cdot \mathbf{n}_e & \text{if } e \subset \Gamma_N, \\ 0 & \text{if } e \subset \Gamma_D, \end{cases}$$

respectively. A global upper bound is then given by [18, Theorem 1]

$$\begin{aligned} \|u - u_h\|_{\text{SUPG}}^2 &\leq \eta_1^2 + \eta_2^2 + \eta_3^2 + \sum_{K \in \mathcal{T}_h} 16\delta_K h_K^{-2} \varepsilon^2 C^2 \|\nabla(u - \tilde{I}_h u_h)\|_K^2 \\ &\quad + \sum_{K \in \mathcal{T}_h} 8\delta_K \varepsilon^2 \|\Delta(u - \tilde{I}_h u_h)\|_K^2, \end{aligned} \quad (2.9)$$

where $\|u - u_h\|_{\text{SUPG}}^2 = \|u - u_h\|_\varepsilon^2 + \sum_{K \in \mathcal{T}_h} \delta_K \|\mathbf{a} \cdot \nabla(u - u_h)\|_K^2$ and \tilde{I}_h is an interpolation operator satisfying the hypothesis in [18]. In the convection-dominated regime, the last two terms in (2.9) are negligible compared with the other terms. The upper bound is reduced to

$$\|u - u_h\|_{\text{SUPG}}^2 \preceq \eta_1^2 + \eta_2^2 + \eta_3^2.$$

Compared with the estimator in [15], one concludes that

$$|||u - u_h|||_*^2 \preceq \eta_1^2 + \eta_2^2 + \eta_3^2 + \text{h.o.t.}$$

On the other hand, when convection dominates, the local lower bound is [18, Theorem 2]

$$\eta_i \preceq \|u - u_h\|_{\text{SUPG}} + \text{h.o.t.}, \quad i = 1, 2, 3.$$

This leads to

$$|||u - u_h|||_* \preceq \|u - u_h\|_{\text{SUPG}} + \text{h.o.t.}$$

Let $|||u - u_h|||_\varepsilon := \|u - u_h\|_\varepsilon + \|h^{1/2} \nabla(u - u_h)\|$. Since

$$\|u - u_h\|_{\text{SUPG}} \preceq |||u - u_h|||_\varepsilon \leq |||u - u_h||| \preceq |||u - u_h|||_*,$$

$|||u - u_h|||_*$ is equivalent to $|||u - u_h|||_\varepsilon$ and $\|u - u_h\|_{\text{SUPG}}$ when the higher order terms are negligible. This will be confirmed numerically in Section 7.

3 Flux Recovery

Introducing the flux variable $\sigma = -\varepsilon \nabla u$, the variational form of the flux reads: find $\sigma \in H(\text{div}; \Omega)$ such that

$$(\varepsilon^{-1} \sigma, \tau) = -(\nabla u, \tau) \quad \forall \tau \in H(\text{div}; \Omega). \quad (3.1)$$

In this paper, we use standard RT_0 or BDM_1 elements to recover the flux, which are

$$RT_0 := \{\tau \in H(\text{div}; \Omega) : \tau|_K \in P_0(K)^d + \mathbf{x}P_0(K) \quad \forall K \in \mathcal{T}_h\}$$

and

$$BDM_1 := \{\tau \in H(\text{div}; \Omega) : \tau|_K \in P_1(K)^d \quad \forall K \in \mathcal{T}_h\},$$

respectively. Let u_h be the solution to (2.4) and \mathcal{V} be RT_0 or BDM_1 . We recover the flux by solving the following problem: find $\sigma_\nu \in \mathcal{V}$ such that

$$(\varepsilon^{-1} \sigma_\nu, \tau) = -(\nabla u_h, \tau) \quad \forall \tau \in \mathcal{V}. \quad (3.2)$$

We have the following *a priori* error estimates for the recovered flux.

Theorem 3.1. *Let u , u_h , σ , and σ_ν be solutions to (2.2), (2.4), (3.1), and (3.2), respectively. Then there holds*

$$\|\varepsilon^{-1/2}(\sigma - \sigma_\nu)\| \leq \inf_{\tau \in \mathcal{V}} \|\varepsilon^{-1/2}(\sigma - \tau)\| + \|\varepsilon^{1/2} \nabla(u - u_h)\|.$$

Proof. Following the line of the proof of [9, Theorem 3.1], we obtain the assertion. \square

We next consider an explicit approximation of the flux in RT_0 (cf., e.g. [9]). For $e \in \mathcal{E}_D \cup \mathcal{E}_N$, let \mathbf{n}_e be the outward unit normal vector to Γ . For $e \in \mathcal{E}_\Omega$, let K_e^+ and K_e^- be the two elements sharing e , and let \mathbf{n}_e be the outward unit normal vector of K_e^+ . Let \mathbf{a}_e^\pm be the opposite vertices of e in K_e^\pm , respectively. Then the RT_0 basis function corresponding to e is

$$\phi_e(\mathbf{x}) := \begin{cases} \frac{|e|}{d|K_e^+|}(\mathbf{x} - \mathbf{a}_e^+) & \text{for } \mathbf{x} \in K_e^+, \\ -\frac{|e|}{d|K_e^-|}(\mathbf{x} - \mathbf{a}_e^-) & \text{for } \mathbf{x} \in K_e^-, \\ 0 & \text{elsewhere,} \end{cases}$$

where $|e|$ and $|K_e^\pm|$ are the $(d-1)$ - and d -dimensional measures of e and K_e^\pm , respectively. For a boundary edge/face e , the corresponding basis function is

$$\phi_e(\mathbf{x}) := \begin{cases} \frac{|e|}{d|K_e^+|}(\mathbf{x} - \mathbf{a}_e^+) & \text{for } \mathbf{x} \in K_e^+, \\ 0 & \text{elsewhere.} \end{cases}$$

Define the approximation $\hat{\sigma}_{RT_0}(u_h)$ of $\boldsymbol{\tau} = -\varepsilon \nabla u_h$ in RT_0 by

$$\hat{\sigma}_{RT_0}(u_h) = \sum_{e \in \mathcal{E}} \hat{\sigma}_e \phi_e(\mathbf{x}), \quad (3.3)$$

where $\hat{\sigma}_e$ is the normal component of $\hat{\sigma}_{RT_0}$ on $e \in \mathcal{E}$ defined by

$$\hat{\sigma}_e := \begin{cases} \gamma_e(\boldsymbol{\tau}|_{K_e^+} \cdot \mathbf{n}_e)|_e + (1 - \gamma_e)(\boldsymbol{\tau}|_{K_e^-} \cdot \mathbf{n}_e)|_e & \text{for } e \in \mathcal{E}_\Omega, \\ (\boldsymbol{\tau}|_{K_e^+} \cdot \mathbf{n}_e)|_e & \text{for } e \in \mathcal{E}_D \cup \mathcal{E}_N, \end{cases} \quad (3.4)$$

with the constant $\gamma_e \in [0, 1]$ to be determined in (5.5). Note that the definition of $\hat{\sigma}_{RT_0}(u_h)$ is independent of the choice of K_e^+ and K_e^- .

4 *A posteriori* Error Estimates

For $K \in \mathcal{T}_h$ and $e \in \mathcal{E}$, define weights $\alpha_K := \min\{h_K \varepsilon^{-1/2}, \beta^{-1/2}, h_K^{1/2}\}$ and $\alpha_e := \min\{h_e^{1/2} \varepsilon^{-1/2}, \varepsilon^{-1/4} \beta^{-1/4}, 1\}$, and residuals

$$R_K := f + \varepsilon \Delta u_h - \mathbf{a} \cdot \nabla u_h - b u_h \quad \text{and} \quad \tilde{R}_K := f - \nabla \cdot \sigma_h - \mathbf{a} \cdot \nabla u_h - b u_h, \quad (4.1)$$

where σ_h is the implicit or explicit recovered flux. Let

$$\Phi = \left(\sum_{K \in \mathcal{T}_h} \alpha_K^2 (\|R_K\|_K^2 + \|\tilde{R}_K\|_K^2) + \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_h\|^2 \right)^{1/2}. \quad (4.2)$$

We have the following error estimates.

Theorem 4.1. Let u and u_h be the solutions to (2.2) and (2.4), respectively. Let Φ be defined in (4.2). If $\sigma_h = \hat{\sigma}_{RT_0}(u_h)$ is the recovered flux obtained by the explicit approximation (3.3), then

$$|||u - u_h|||_* \preceq \Phi + \left(\sum_{e \in \Gamma_N} \alpha_e^2 \|g - \varepsilon \nabla u_h \cdot \mathbf{n}\|_e^2 \right)^{1/2}. \quad (4.3)$$

If $\sigma_h = \sigma_\nu$ is the recovered flux obtained by the implicit approximation (3.2), then

$$|||u - u_h|||_* \preceq \Phi + \left(\sum_{e \in \Gamma_N} \alpha_e^2 (\|g - \varepsilon \nabla u_h \cdot \mathbf{n}\|_e^2 + \|(\sigma_h + \varepsilon \nabla u_h) \cdot \mathbf{n}\|_e^2) \right)^{1/2}. \quad (4.4)$$

Proof. For any $v \in H_D^1(\Omega)$, let $I_h v$ be the Cl  ment interpolation of v . Using (2.2), and integration by parts, we have

$$\begin{aligned} B(u - u_h, v) &= (f - \mathbf{a} \cdot \nabla u_h - b u_h, v) - (\varepsilon \nabla u_h, \nabla v) + \langle g, v \rangle_{\Gamma_N} \\ &= (\tilde{R}_K, v) - (\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_h, \varepsilon^{1/2} \nabla v) + \langle g + \sigma_h \cdot \mathbf{n}, v \rangle_{\Gamma_N}, \end{aligned}$$

which implies

$$\begin{aligned} B(u - u_h, v - I_h v) &= - \sum_{K \in \mathcal{T}_h} (\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_h, \varepsilon^{1/2} \nabla (v - I_h v))_K \\ &\quad + \sum_{K \in \mathcal{T}_h} (\tilde{R}_K, v - I_h v)_K + \sum_{e \in \Gamma_N} \langle g + \sigma_h \cdot \mathbf{n}, v - I_h v \rangle_e. \end{aligned} \quad (4.5)$$

Subtracting (2.4) from (2.2), we get

$$B(u - u_h, I_h v) = - \sum_{K \in \mathcal{T}_h} \delta_K (R_K, \mathbf{a} \cdot \nabla (I_h v))_K. \quad (4.6)$$

On the other hand, the Cl  ment interpolation operator possesses the following stable estimate (cf. [12, Exercise 3.2.3] and [13, 28, 25])

$$\|\nabla(v - I_h v)\|_K \preceq \|\nabla v\|_{\tilde{\omega}_K} \quad \forall K \in \mathcal{T}_h, \quad v \in H^1(\tilde{\omega}_K),$$

where $\tilde{\omega}_K$ is the union of all elements in \mathcal{T}_h sharing at least one point with K . Then from (4.5), (4.6), and Lemma 2.1, we obtain

$$\begin{aligned} B(u - u_h, v) &= B(u - u_h, v - I_h v) + B(u - u_h, I_h v) \\ &\preceq \left(\sum_{K \in \mathcal{T}_h} \max\{\beta, \varepsilon h_K^{-2}, h_K^{-1}\}^{-1} (\|R_K\|_K^2 + \|\tilde{R}_K\|_K^2) + \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_h\|^2 \right. \\ &\quad \left. + \sum_{e \in \Gamma_N} \max\{\varepsilon^{1/2} \beta^{1/2}, \varepsilon h_e^{-1}, 1\}^{-1} \|g + \sigma_h \cdot \mathbf{n}\|_e^2 \right)^{1/2} |||v|||. \end{aligned} \quad (4.7)$$

If σ_h is the recovery flux obtained by its explicit approximation (3.3), i.e., $\sigma_h = \hat{\sigma}_{RT_0}(u_h)$, then we have from the construction of $\hat{\sigma}_{RT_0}(u_h)$ that

$$\sigma_h \cdot \mathbf{n} = -\varepsilon \nabla u_h \cdot \mathbf{n} \quad \text{on } \Gamma_N.$$

Thus (4.3) follows from (4.7). If σ_h is the recovery flux obtained by the implicit approximation (3.2), i.e., $\sigma_h = \sigma_\nu$, then (4.4) follows from a triangle inequality and (4.7). \square

Remark 4.2. Compared with the estimators developed in [15], the jump in the normal component of the flux is replaced by the residual $\|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_h\|$ in Theorem 4.1. Moreover, a residual term $\sum_{K \in \mathcal{T}_h} \alpha_K^2 \|\tilde{R}_K\|_K^2$ and another residual term of the Neumann boundary data occur in the a posteriori error estimators. In $\|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_h\|$, the impacts of ε and h are implicitly accounted, which however are expressed explicitly in α_e , and hence in $(\sum_{e \in \mathcal{E}_h} \alpha_e^2 \|R_e\|_e^2)^{1/2}$, in [15] (e.g., if $\varepsilon \leq h_e$ and $\beta = 1$, then $\alpha_e = 1$). To illustrate the difference in numerical results, we provide in Figure 1 the adaptive meshes by the two estimators for Section 7 Example 1. It is observed that the quality of the meshes generated by $\|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_h\|$ is better than that of the meshes by $(\sum_{e \in \mathcal{E}_h} \alpha_e^2 \|R_e\|_e^2)^{1/2}$.

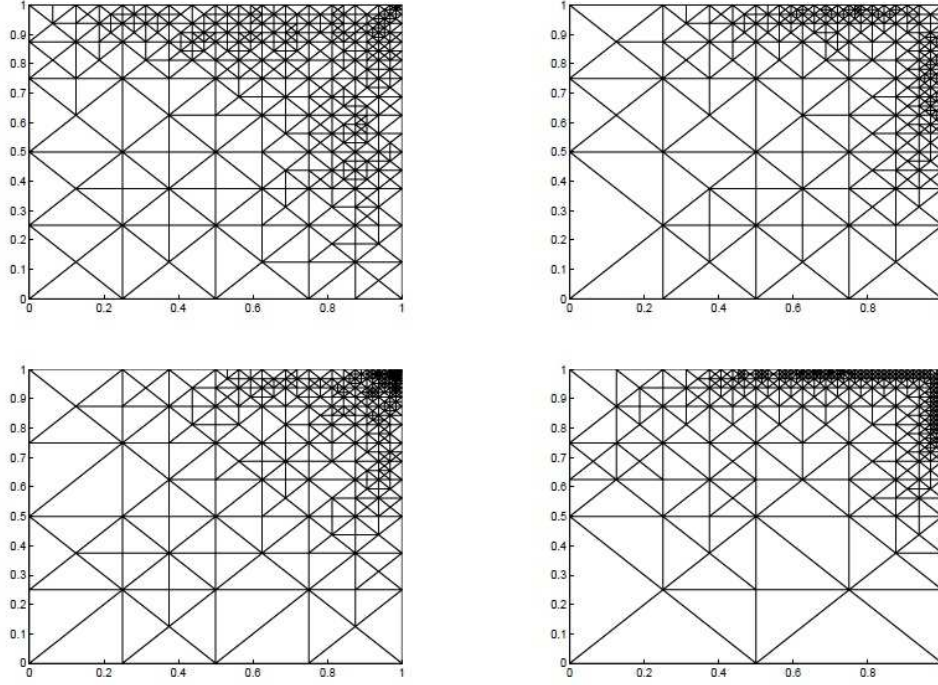


Figure 1: Top: the meshes by using $\varepsilon = 0.01$, $\delta_K = h_K$, and $\theta = 0.3$; Bottom: the meshes by using $\varepsilon = 0.0001$, $\delta_K = h_K$, and $\theta = 0.5$; Left: the meshes by using $(\sum_{e \in \mathcal{E}_h} \alpha_e^2 \|R_e\|_e^2)^{1/2}$; and Right: the meshes by using $\|\varepsilon^{-1/2} \sigma_h + \varepsilon^{1/2} \nabla u_h\|$ with σ_h the implicit recovery flux by (3.2). The top-left, top-right, bottom-left, and bottom-right plots are meshes after 10, 8, 8, and 8 iterations with 640, 473, 635, and 749 triangles, respectively.

5 Analysis of efficiency

Let $\tau = -\varepsilon \nabla u_h$. For each $e \in \mathcal{E}_\Omega$, define the edge/face residual along e by

$$R_e := \begin{cases} J_e(\tau) & \text{if } e \not\subset \Gamma, \\ g + \tau \cdot \mathbf{n}_e & \text{if } e \subset \Gamma_N, \\ 0 & \text{if } e \subset \Gamma_D, \end{cases}$$

where $J_e(\tau)$ is defined in (5.3). Let Π_k be an L^2 -projection operator into $P_k(K)$ and

$$\text{osc}_h := \left(\sum_{K \in \mathcal{T}_h} \alpha_K^2 \|D_K\|_K^2 + \sum_{e \in \Gamma_N} \alpha_e^2 \|D_e\|_e^2 \right)^{1/2},$$

be an oscillation of data, where $D_K = R_K - \Pi_k R_K$ for every $K \in \mathcal{T}_h$, and $D_e = R_e - \Pi_k R_e$ for each $e \subset \Gamma_N$. The following efficient estimate is found in [15].

Lemma 5.1. *Let u and u_h be the solutions to problems (2.2) and (2.4), respectively. Then the error is bounded from below by*

$$\left(\sum_{K \in \mathcal{T}_h} (\alpha_K^2 \|R_K\|_K^2 + \sum_{e \subset \partial K} \alpha_e^2 \|R_e\|_e^2) \right)^{1/2} \preceq \|u - u_h\|_* + \text{osc}_h. \quad (5.1)$$

Lemma 5.2. *Let u and u_h be the solutions to (2.2) and (2.4), respectively. If $\hat{\sigma}_{RT_0}(u_h)$ is the explicit recovery flux given by (3.3), then it holds*

$$\|\varepsilon^{-1/2} \hat{\sigma}_{RT_0}(u_h) + \varepsilon^{1/2} \nabla u_h\| \preceq \|u - u_h\|_* + \text{osc}_h. \quad (5.2)$$

Proof. For any element $K \in \mathcal{T}_h$ and an edge/face $e \subset \partial K$, let \mathbf{n}_e be the outward unit vector normal to ∂K . Note that $\tau = -\varepsilon \nabla u_h$ on K is a constant vector. Let $\tau_{e,K} = (\tau|_K \cdot \mathbf{n}_e)|_e$ be the normal component of τ on e . There holds the representation in RT_0 : $\tau = \sum_{e \subset \partial K} \tau_{e,K} \phi_e(\mathbf{x})$. Then, for $\mathbf{x} \in K$, (3.3) and (3.4) give

$$\hat{\sigma}_{RT_0}(u_h) - \tau = \sum_{e \subset \partial K \cap \mathcal{E}_\Omega} (\hat{\sigma}_e - \tau|_e \cdot \mathbf{n}_e) \phi_e(\mathbf{x}) = \sum_{e \subset \partial K \setminus \Gamma} (1 - \gamma_e) J_e(\tau) \phi_e(\mathbf{x}),$$

where, for the two elements K_e^+ and K_e^- sharing e ,

$$J_e(\boldsymbol{\tau}) = (\boldsymbol{\tau}|_{K_e^+} - \boldsymbol{\tau}|_{K_e^-}) \cdot \mathbf{n}_e. \quad (5.3)$$

This identity implies

$$\begin{aligned} \|\varepsilon^{-1/2}(\hat{\sigma}_{RT_0}(u_h) - \boldsymbol{\tau})\|_K^2 &\preceq \sum_{e \subset \partial K \setminus \Gamma} \frac{(1 - \gamma_e)^2}{\varepsilon} \|J_e(\boldsymbol{\tau}) \phi_e(\mathbf{x})\|_K^2 \\ &\leq \sum_{e \subset \partial K \setminus \Gamma} \frac{(1 - \gamma_e)^2}{\varepsilon} |J_e(\boldsymbol{\tau})|^2 \|\phi_e(\mathbf{x})\|_K^2 \preceq \sum_{e \subset \partial K \setminus \Gamma} \frac{(1 - \gamma_e)^2}{\varepsilon} \|J_e(\boldsymbol{\tau})\|_e^2 h_e, \end{aligned} \quad (5.4)$$

where, in the last step, we employ the fact that $J_e(\boldsymbol{\tau})$ is constant and $\|\phi_e(\mathbf{x})\|_K^2 \preceq |K|$.

Now, for each $e \in \mathcal{E}_\Omega$ we choose

$$\gamma_e = 1 - \alpha_e \varepsilon^{1/2} h_e^{-1/2} \quad (5.5)$$

so that $(1 - \gamma_e)^2 h_e / \varepsilon = 1$. Since $\alpha_e \leq \sqrt{h_e / \varepsilon}$, thus $0 \leq \gamma_e < 1$. This choice together with the definition of the edge/face residual R_e leads to

$$\frac{(1 - \gamma_e)^2}{\varepsilon} h_e \|J_e(\boldsymbol{\tau})\|_e^2 \leq \alpha_e^2 \|R_e\|_e^2,$$

which, with (5.4), implies

$$\|\varepsilon^{-1/2} \hat{\sigma}_{RT_0}(u_h) + \varepsilon^{1/2} \nabla u_h\|_K^2 \preceq \sum_{e \subset \partial K \setminus \Gamma} \alpha_e^2 \|R_e\|_e^2. \quad (5.6)$$

Summing up (5.6) over all $K \in \mathcal{T}_h$, we obtain

$$\|\varepsilon^{-1/2} \hat{\sigma}_{RT_0}(u_h) + \varepsilon^{1/2} \nabla u_h\|^2 \preceq \sum_{K \in \mathcal{T}_h} \sum_{e \subset \partial K \setminus \Gamma} \alpha_e^2 \|R_e\|_e^2 \preceq \sum_{K \in \mathcal{T}_h} \sum_{e \subset \partial K} \alpha_e^2 \|R_e\|_e^2. \quad (5.7)$$

The desired estimate (5.2) follows from (5.7) and (5.1). \square

Lemma 5.3. *Under the assumption of Lemma 5.2, if σ_ν is the implicit recovery flux obtained by (3.2), then it holds*

$$\|\varepsilon^{-1/2} \sigma_\nu + \varepsilon^{1/2} \nabla u_h\| \preceq \|u - u_h\|_* + \text{osc}_h. \quad (5.8)$$

Proof. For all $\tau \in \mathcal{V}$, (3.2) implies

$$(\varepsilon^{-1/2} \sigma_\nu + \varepsilon^{1/2} \nabla u_h, \varepsilon^{-1/2} \tau) = 0,$$

which results in

$$\begin{aligned} \|\varepsilon^{-1/2} \sigma_\nu + \varepsilon^{1/2} \nabla u_h\|^2 &= (\varepsilon^{-\frac{1}{2}} \sigma_\nu + \varepsilon^{\frac{1}{2}} \nabla u_h, \varepsilon^{-\frac{1}{2}} (\sigma_\nu - \tau) + \varepsilon^{-\frac{1}{2}} \tau + \varepsilon^{\frac{1}{2}} \nabla u_h) \\ &= (\varepsilon^{-1/2} \sigma_\nu + \varepsilon^{1/2} \nabla u_h, \varepsilon^{-1/2} \tau + \varepsilon^{1/2} \nabla u_h) \leq \|\varepsilon^{-1/2} \sigma_\nu + \varepsilon^{1/2} \nabla u_h\| \|\varepsilon^{-1/2} \tau + \varepsilon^{1/2} \nabla u_h\|. \end{aligned}$$

Dividing by $\|\varepsilon^{-1/2} \sigma_\nu + \varepsilon^{1/2} \nabla u_h\|$, we get

$$\|\varepsilon^{-1/2} \sigma_\nu + \varepsilon^{1/2} \nabla u_h\| \leq \|\varepsilon^{-1/2} \tau + \varepsilon^{1/2} \nabla u_h\|$$

for all $\tau \in \mathcal{V}$, which implies

$$\|\varepsilon^{-1/2} \sigma_\nu + \varepsilon^{1/2} \nabla u_h\| = \min_{\tau \in \mathcal{V}} \|\varepsilon^{-1/2} \tau + \varepsilon^{1/2} \nabla u_h\|. \quad (5.9)$$

The assertion (5.8) follows from the fact that $RT_0 \subset BDM_1$, (5.9), and Lemma 5.2. \square

Lemma 5.4. *Under the assumption of Lemma 5.2, if σ_ν is the implicit recovery flux obtained by (3.2), then it holds*

$$\left(\sum_{e \subset \Gamma_N} \alpha_e^2 \|(\sigma_\nu + \varepsilon \nabla u_h) \cdot \mathbf{n}\|_e^2 \right)^{1/2} \preceq \|\varepsilon^{-1/2} \sigma_\nu + \varepsilon^{1/2} \nabla u_h\|. \quad (5.10)$$

Proof. Using trace theorem, inverse estimate, shape regularity of element, and the fact $\alpha_e \leq h_e^{1/2} / \sqrt{\varepsilon}$, we have for $e \subset \Gamma_N \cap \partial K$

$$\alpha_e \|(\sigma_\nu + \varepsilon \nabla u_h) \cdot \mathbf{n}\|_e \leq \alpha_e h_K^{-1/2} \|\sigma_\nu + \varepsilon \nabla u_h\|_K \preceq \|\varepsilon^{-1/2} \sigma_\nu + \varepsilon^{1/2} \nabla u_h\|_K.$$

Summing the above inequality over all $e \subset \Gamma_N$, we obtain the desired estimate (5.10). \square

Moreover, we have the following estimate.

Lemma 5.5. *Let σ_h be the flux recovery obtained by the implicit approximation (3.2) or the explicit approximation (3.3). Then it holds*

$$\left(\sum_{K \in \mathcal{T}_h} \alpha_K^2 \|\tilde{R}_K\|_K^2 \right)^{1/2} \preceq \left(\sum_{K \in \mathcal{T}_h} \alpha_K^2 \|R_K\|_K^2 \right)^{1/2} + \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_h\|. \quad (5.11)$$

Proof. For each $K \in \mathcal{T}_h$, it follows from triangle inequality and inverse estimate that

$$\begin{aligned} \|\tilde{R}_K\|_K &\leq \|R_K\|_K + \|\varepsilon \Delta u_h + \nabla \cdot \sigma_h\|_K \\ &\preceq \|R_K\|_K + h_K^{-1} \varepsilon^{1/2} \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_h\|_K. \end{aligned}$$

We get from the fact $\alpha_K \leq h_K / \sqrt{\varepsilon}$ that

$$\alpha_K \|\tilde{R}_K\|_K \preceq \alpha_K \|R_K\|_K + \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_h\|_K.$$

Summing up the above inequality over all $K \in \mathcal{T}_h$, we obtain

$$\sum_{K \in \mathcal{T}_h} \alpha_K^2 \|\tilde{R}_K\|_K^2 \preceq \sum_{K \in \mathcal{T}_h} \alpha_K^2 \|R_K\|_K^2 + \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_h\|^2,$$

which results in the desired estimate (5.11). \square

Collecting Lemma 5.1-5.5, we obtain the global lower bound estimate.

Theorem 5.6. *Let u and u_h be the solutions to (2.2) and (2.4), respectively. Let Φ be defined in (4.2). If σ_h is the recovery flux obtained by the explicit approximation (3.3), i.e., $\sigma_h = \hat{\sigma}_{RT_0}(u_h)$, then*

$$\Phi + \left(\sum_{e \in \Gamma_N} \alpha_e^2 \|g - \varepsilon \nabla u_h \cdot \mathbf{n}\|_e^2 \right)^{1/2} \preceq \|u - u_h\|_* + \text{osc}_h. \quad (5.12)$$

If σ_h is the recovery flux obtained by the implicit approximation (3.2), i.e., $\sigma_h = \sigma_\nu$, then

$$\Phi + \left(\sum_{e \in \Gamma_N} \alpha_e^2 (\|g - \varepsilon \nabla u_h \cdot \mathbf{n}\|_e^2 + \|(\sigma_h + \varepsilon \nabla u_h) \cdot \mathbf{n}\|_e^2) \right)^{1/2} \preceq \|u - u_h\|_* + \text{osc}_h. \quad (5.13)$$

Proof. It follows from Lemmas 5.2-5.3 that

$$\|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_h\|^2 \preceq \|u - u_h\|_*^2 + \text{osc}_h. \quad (5.14)$$

If $\sigma_h = \hat{\sigma}_{RT_0}(u_h)$, we get from Lemma 5.1 and Lemma 5.5 that

$$\begin{aligned} &\sum_{K \in \mathcal{T}_h} \alpha_K^2 (\|R_K\|_K^2 + \|\tilde{R}_K\|_K^2) + \sum_{e \in \Gamma_N} \alpha_e^2 \|g - \varepsilon \nabla u_h \cdot \mathbf{n}\|_e^2 \\ &\leq \sum_{K \in \mathcal{T}_h} (\alpha_K^2 \|R_K\|_K^2 + \sum_{e \subset \partial K} \alpha_e^2 \|R_e\|_e^2) + \sum_{K \in \mathcal{T}_h} \alpha_K^2 \|\tilde{R}_K\|_K^2 \\ &\preceq \sum_{K \in \mathcal{T}_h} (\alpha_K^2 \|R_K\|_K^2 + \sum_{e \subset \partial K} \alpha_e^2 \|R_e\|_e^2) + \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_h\|^2 \\ &\preceq \|u - u_h\|_*^2 + \text{osc}_h^2 + \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_h\|^2. \end{aligned} \quad (5.15)$$

The assertion (5.12) follows from a combination of (5.14) and (5.15). If $\sigma_h = \sigma_\nu$, then, similarly, we have from Lemma 5.1 and Lemmas 5.4-5.5 that

$$\begin{aligned} &\sum_{K \in \mathcal{T}_h} \alpha_K^2 (\|R_K\|_K^2 + \|\tilde{R}_K\|_K^2) + \sum_{e \in \Gamma_N} \alpha_e^2 (\|g - \varepsilon \nabla u_h \cdot \mathbf{n}\|_e^2 + \|(\sigma_h + \varepsilon \nabla u_h) \cdot \mathbf{n}\|_e^2) \\ &\preceq \|u - u_h\|_*^2 + \text{osc}_h^2 + \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_h\|^2. \end{aligned}$$

The estimate (5.13) follows from the above inequality and (5.14). \square

6 A stabilization $H(\text{div})$ recovery

Let $u_h \in V_h$ be the approximation of the solution u to (1.1). A stabilization $H(\text{div})$ recovery procedure is to find $\sigma_{\mathcal{T}} \in \mathcal{V}$ such that

$$\begin{aligned} & (\varepsilon^{-1}\sigma_{\mathcal{T}}, \tau_{\nu}) + \sum_{K \in \mathcal{T}_h} \gamma_K (\nabla \cdot \sigma_{\mathcal{T}}, \nabla \cdot \tau_{\nu})_K \\ &= -(\nabla u_h, \tau_{\nu}) + \sum_{K \in \mathcal{T}_h} \gamma_K (f - \mathbf{a} \cdot \nabla u_h - bu_h, \nabla \cdot \tau_{\nu})_K \quad \forall \tau_{\nu} \in \mathcal{V}, \end{aligned} \quad (6.1)$$

where γ_K is a stabilization parameter to be determined in below. Recalling the exact flux $\sigma = -\varepsilon \nabla u$, define the approximation error of the flux recovery by

$$\|\sigma - \sigma_{\mathcal{T}}\|_{B,\Omega}^2 := (\varepsilon^{-1}(\sigma - \sigma_{\mathcal{T}}), \sigma - \sigma_{\mathcal{T}}) + \sum_{K \in \mathcal{T}_h} \gamma_K (\nabla \cdot (\sigma - \sigma_{\mathcal{T}}), \nabla \cdot (\sigma - \sigma_{\mathcal{T}}))_K.$$

Theorem 6.1. *The following a priori error bound for the approximation error of the $H(\text{div})$ recovery flux holds*

$$\|\sigma - \sigma_{\mathcal{T}}\|_{B,\Omega} \preceq \inf_{\tau_{\nu} \in \mathcal{V}} \|\sigma - \tau_{\nu}\|_{B,\Omega} + \|u - u_h\|_{\varepsilon}. \quad (6.2)$$

Proof. Note that the exact flux σ satisfies, for all $\tau \in H(\text{div}; \Omega)$,

$$(\varepsilon^{-1}\sigma, \tau) + \sum_{K \in \mathcal{T}_h} \gamma_K (\nabla \cdot \sigma, \nabla \cdot \tau)_K = -(\nabla u, \tau) + \sum_{K \in \mathcal{T}_h} \gamma_K (f - \mathbf{a} \cdot \nabla u - bu, \nabla \cdot \tau)_K.$$

For all $\tau_{\nu} \in \mathcal{V}$, This identity and (6.1) give the error equation

$$\begin{aligned} & (\varepsilon^{-1}(\sigma - \sigma_{\mathcal{T}}), \tau_{\nu}) + \sum_{K \in \mathcal{T}_h} \gamma_K (\nabla \cdot (\sigma - \sigma_{\mathcal{T}}), \nabla \cdot \tau_{\nu})_K \\ &= -(\nabla(u - u_h), \tau_{\nu}) - \sum_{K \in \mathcal{T}_h} \gamma_K (\mathbf{a} \cdot \nabla(u - u_h) + b(u - u_h), \nabla \cdot \tau_{\nu})_K \end{aligned} \quad (6.3)$$

which implies

$$\begin{aligned} \|\sigma - \sigma_{\mathcal{T}}\|_{B,\Omega}^2 &= (\varepsilon^{-1}(\sigma - \sigma_{\mathcal{T}}), \sigma - \tau_{\nu}) + \sum_{K \in \mathcal{T}_h} \gamma_K (\nabla \cdot (\sigma - \sigma_{\mathcal{T}}), \nabla \cdot (\sigma - \tau_{\nu}))_K \\ &\quad - (\nabla(u - u_h), \tau_{\nu} - \sigma_{\mathcal{T}}) - \sum_{K \in \mathcal{T}_h} \gamma_K (\mathbf{a} \cdot \nabla(u - u_h) + b(u - u_h), \nabla \cdot (\tau_{\nu} - \sigma_{\mathcal{T}}))_K. \end{aligned}$$

Using (1.2) and Cauchy-Schwartz inequality, we arrive at

$$\begin{aligned} \|\sigma - \sigma_{\mathcal{T}}\|_{B,\Omega}^2 &\leq \|\sigma - \sigma_{\mathcal{T}}\|_{B,\Omega} \|\sigma - \tau_{\nu}\|_{B,\Omega} + \|u - u_h\|_{\varepsilon} \|\tau_{\nu} - \sigma_{\mathcal{T}}\|_{B,\Omega} \\ &\quad + \sum_{K \in \mathcal{T}_h} \gamma_K (\|\mathbf{a}\|_{L^{\infty}(K)} \|\nabla(u - u_h)\|_K + c_b \beta \|u - u_h\|_K) \|\nabla \cdot (\tau_{\nu} - \sigma_{\mathcal{T}})\|_K. \end{aligned}$$

Choose $\gamma_K \leq h_K \min \left\{ \frac{1}{\|\mathbf{a}\|_{L^{\infty}(K)}}, \frac{1}{\sqrt{\beta\varepsilon}} \right\}$ for all $K \in \mathcal{T}_h$. Then, by inverse estimate, we have

$$\begin{aligned} \|\sigma - \sigma_{\mathcal{T}}\|_{B,\Omega}^2 &\preceq \|\sigma - \sigma_{\mathcal{T}}\|_{B,\Omega} \|\sigma - \tau_{\nu}\|_{B,\Omega} + \|u - u_h\|_{\varepsilon} \|\tau_{\nu} - \sigma_{\mathcal{T}}\|_{B,\Omega} \\ &\leq \|\sigma - \sigma_{\mathcal{T}}\|_{B,\Omega} \|\sigma - \tau_{\nu}\|_{B,\Omega} + \|u - u_h\|_{\varepsilon} (\|\tau_{\nu} - \sigma\|_{B,\Omega} + \|\sigma - \sigma_{\mathcal{T}}\|_{B,\Omega}), \end{aligned}$$

which implies

$$\|\sigma - \sigma_{\mathcal{T}}\|_{B,\Omega} \preceq \|\sigma - \tau_{\nu}\|_{B,\Omega} + \|u - u_h\|_{\varepsilon} \quad \forall \tau_{\nu} \in \mathcal{V}.$$

The assertion (6.2) follows immediately. \square

Theorem 6.2. *Let $\sigma_{\mathcal{T}}$ be the $H(\text{div})$ recovery flux obtained from (6.1), and u and u_h be the solutions to (2.2) and (2.4), respectively. For each $K \in \mathcal{T}_h$, let $\tilde{R}_K := f - \nabla \cdot \sigma_{\mathcal{T}} - \mathbf{a} \cdot \nabla u_h - bu_h$. Then the following reliable estimate holds*

$$\begin{aligned} \|u - u_h\|_* &\preceq \left(\sum_{K \in \mathcal{T}_h} \alpha_K^2 (\|R_K\|_K^2 + \|\tilde{R}_K\|_K^2) + \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_{\mathcal{T}}\|^2 \right)^{1/2} \\ &\quad + \left(\sum_{e \in \Gamma_N} \alpha_e^2 (\|g - \varepsilon \nabla u_h \cdot \mathbf{n}\|_e^2 + \|(\sigma_{\mathcal{T}} + \varepsilon \nabla u_h) \cdot \mathbf{n}\|_e^2) \right)^{1/2}. \end{aligned} \quad (6.4)$$

Proof. Following the line of the proof of Theorem 4.1, we obtain the estimate (6.4). \square

Lemma 6.3. *Under the assumption of Lemma 5.2, if $\sigma_{\mathcal{T}}$ is the $H(\text{div})$ recovery flux obtained from (6.1), then it holds that*

$$\left(\sum_{e \in \Gamma_N} \alpha_e^2 \|(\sigma_{\mathcal{T}} + \varepsilon \nabla u_h) \cdot \mathbf{n}\|_e^2 \right)^{1/2} \preceq \|\varepsilon^{-1/2} \sigma_{\mathcal{T}} + \varepsilon^{1/2} \nabla u_h\|. \quad (6.5)$$

Proof. A proof similar to Lemma 5.4 yields the assertion (6.5). \square

Lemma 6.4. *Let $\sigma_{\mathcal{T}}$ be the $H(\text{div})$ recovery flux obtained from (6.1), and \tilde{R}_K be the residual defined in Theorem 6.2. Then it holds that*

$$\left(\sum_{K \in \mathcal{T}_h} \alpha_K^2 \|\tilde{R}_K\|_K^2 \right)^{1/2} \preceq \left(\sum_{K \in \mathcal{T}_h} \alpha_K^2 \|R_K\|_K^2 \right)^{1/2} + \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_{\mathcal{T}}\|. \quad (6.6)$$

Proof. Following the line of the proof of Lemma 5.5, we obtain the assertion (6.6). \square

Lemma 6.5. *Let u and u_h be the solutions to (2.2) and (2.4), respectively, and $\sigma_{\mathcal{T}}$ be the $H(\text{div})$ recovery flux obtained from (6.1). Then it holds that*

$$\|\varepsilon^{-1/2} \sigma_{\mathcal{T}} + \varepsilon^{1/2} \nabla u_h\| \preceq \|u - u_h\|_* + \text{osc}_h. \quad (6.7)$$

Proof. For all $\tau_\nu \in \mathcal{V}$, we have from (6.1) that

$$\begin{aligned} \|\varepsilon^{-1/2} \sigma_{\mathcal{T}} + \varepsilon^{1/2} \nabla u_h\|^2 &= (\varepsilon^{-1/2} \sigma_{\mathcal{T}} + \varepsilon^{1/2} \nabla u_h, \varepsilon^{-1/2} \tau_\nu + \varepsilon^{1/2} \nabla u_h) \\ &\quad + (\varepsilon^{-1/2} \sigma_{\mathcal{T}} + \varepsilon^{1/2} \nabla u_h, \varepsilon^{-1/2} (\sigma_{\mathcal{T}} - \tau_\nu)) \\ &= (\varepsilon^{-1/2} \sigma_{\mathcal{T}} + \varepsilon^{1/2} \nabla u_h, \varepsilon^{-1/2} \tau_\nu + \varepsilon^{1/2} \nabla u_h) \\ &\quad + \sum_{K \in \mathcal{T}_h} \gamma_K (f - \mathbf{a} \cdot \nabla u_h - b u_h - \nabla \cdot \sigma_{\mathcal{T}}, \nabla \cdot (\sigma_{\mathcal{T}} - \tau_\nu))_K. \end{aligned} \quad (6.8)$$

An inverse estimate leads to

$$\begin{aligned} &(f - \mathbf{a} \cdot \nabla u_h - b u_h - \nabla \cdot \sigma_{\mathcal{T}}, \nabla \cdot (\sigma_{\mathcal{T}} - \tau_\nu))_K \\ &= (R_K - (\varepsilon \Delta u_h + \nabla \cdot \sigma_{\mathcal{T}}), \nabla \cdot (\sigma_{\mathcal{T}} - \tau_\nu))_K \\ &\leq (\|R_K\|_K + C h_K^{-1} \varepsilon^{1/2} \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_{\mathcal{T}}\|_K) C h_K^{-1} \varepsilon^{1/2} \|\varepsilon^{-1/2} (\sigma_{\mathcal{T}} - \tau_\nu)\|_K. \end{aligned} \quad (6.9)$$

Choose $\gamma_K > 0$ to satisfy

$$\gamma_K \leq h_K \min \left\{ \frac{1}{\|\mathbf{a}\|_{L^\infty(K)}}, \frac{1}{\sqrt{\beta \varepsilon}}, \frac{\alpha_K}{8 C^2 \sqrt{\varepsilon}} \right\} \quad \forall K \in \mathcal{T}_h.$$

From (6.8)-(6.9), Young's inequality, $\alpha_K \leq h_K / \sqrt{\varepsilon}$, and triangle inequality, we have

$$\begin{aligned} \|\varepsilon^{-1/2} \sigma_{\mathcal{T}} + \varepsilon^{1/2} \nabla u_h\|^2 &\leq \frac{1}{8} \|\varepsilon^{-1/2} \sigma_{\mathcal{T}} + \varepsilon^{1/2} \nabla u_h\|^2 + 2 \|\varepsilon^{-1/2} \tau_\nu + \varepsilon^{1/2} \nabla u_h\|^2 \\ &\quad + \sum_{K \in \mathcal{T}_h} \left(\frac{1}{8C} \alpha_K \|R_K\|_K + \frac{1}{8} \|\varepsilon^{-1/2} \sigma_{\mathcal{T}} + \varepsilon^{1/2} \nabla u_h\|_K \right) \|\varepsilon^{-1/2} (\sigma_{\mathcal{T}} - \tau_\nu)\|_K \\ &\leq \frac{3}{8} \|\varepsilon^{-1/2} \sigma_{\mathcal{T}} + \varepsilon^{1/2} \nabla u_h\|^2 + \frac{17}{8} \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \tau_\nu\|^2 + \frac{1}{8C^2} \sum_{K \in \mathcal{T}_h} \alpha_K^2 \|R_K\|_K^2, \end{aligned}$$

which results in, for all $\tau_\nu \in \mathcal{V}$,

$$\|\varepsilon^{-1/2} \sigma_{\mathcal{T}} + \varepsilon^{1/2} \nabla u_h\|^2 \leq \frac{17}{5} \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \tau_\nu\|^2 + \frac{1}{5C^2} \sum_{K \in \mathcal{T}_h} \alpha_K^2 \|R_K\|_K^2.$$

Therefore,

$$\|\varepsilon^{-1/2} \sigma_{\mathcal{T}} + \varepsilon^{1/2} \nabla u_h\|^2 \leq \frac{17}{5} \min_{\tau_\nu \in \mathcal{V}} \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \tau_\nu\|^2 + \frac{1}{5C^2} \sum_{K \in \mathcal{T}_h} \alpha_K^2 \|R_K\|_K^2.$$

By taking τ_ν obtained by the implicit approximation (3.2) or the explicit approximation (3.3), and using the fact that $RT_0 \subset BDM_1$ and Lemmas 5.1-5.3, we obtain the assertion (6.7). \square

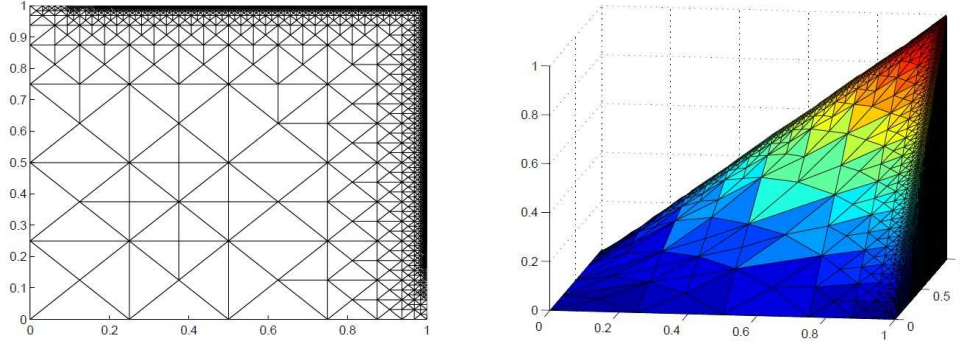


Figure 2: An adaptive mesh with 54855 triangles (left) and the approximation of displacement (piecewise linear element) on the corresponding mesh (right) for $\varepsilon = 10^{-12}$ by using the estimator from (3.3).

Theorem 6.6. *Let u and u_h be the solutions to (2.2) and (2.4), respectively, and $\sigma_{\mathcal{T}}$ be the $H(\text{div})$ recovery flux obtained from (6.1). Then there holds*

$$\begin{aligned} & \left(\sum_{K \in \mathcal{T}_h} \alpha_K^2 (\|R_K\|_K^2 + \|\tilde{R}_K\|_K^2) + \|\varepsilon^{1/2} \nabla u_h + \varepsilon^{-1/2} \sigma_{\mathcal{T}}\|^2 \right)^{1/2} \\ & + \left(\sum_{e \in \Gamma_N} \alpha_e^2 (\|g - \varepsilon \nabla u_h \cdot \mathbf{n}\|_e^2 + \|(\sigma_{\mathcal{T}} + \varepsilon \nabla u_h) \cdot \mathbf{n}\|_e^2) \right)^{1/2} \preceq \|u - u_h\|_* + \text{osc}_h. \end{aligned} \quad (6.10)$$

Proof. Collecting Lemma 5.1, and Lemmas 6.3-6.5, we obtain the estimate (6.10). \square

Remark 6.7 (On three recovering approaches). *First, note that the explicit recovering does not require solving an algebraic system, which, however, is demanded by the implicit and $H(\text{div})$ approaches. From the perspective of accuracy, the implicit and $H(\text{div})$ recoveries are intuitively better than the explicit scheme. L^2 -projection recovery is a special case of $H(\text{div})$ recovering in which the stabilization parameter $\gamma_K = 0$. $H(\text{div})$ recovering is based on mixed FEM, which has been used to precisely approximate the flux [10]. In particular, for L^2 -projection recovery, when $\mathcal{V} = \text{BDM}_1$, following the idea of multipoint flux mixed FEM in [29], one concludes that the cost of solving an algebraic system is equivalent to that for computing the estimator.*

7 Numerical experiments

In this section, we will demonstrate the performance of our *a posteriori* error estimators in two example problems.

7.1 Example 1: boundary layer

In this example, we take $\Omega = (0, 1)^2$, $\mathbf{a} = (1, 1)$, and $b = 1$. We use $\beta = 1$ and set the right-hand side f so that the exact solution of (1.1) is

$$u(x, y) = \left(\frac{\exp(\frac{x-1}{\varepsilon}) - 1}{\exp(-\frac{1}{\varepsilon}) - 1} + x - 1 \right) \left(\frac{\exp(\frac{y-1}{\varepsilon}) - 1}{\exp(-\frac{1}{\varepsilon}) - 1} + y - 1 \right).$$

Clearly, u is 0 on Γ and has boundary layers of width $\mathcal{O}(\varepsilon)$ along $x = 1$ and $y = 1$. Note that for a fixed ε , similar as in [5], one can numerically compute the characteristic layers. However, we shall be focused on numerical robustness of the estimators in this paper.

The coarsest triangulation \mathcal{T}_0 is obtained from halving 4 congruent squares by connecting the bottom right and top left corners. We employ Dörfler strategy with the marking parameter $\theta = 0.5$, and use the “longest edge” refinement to obtain an admissible mesh.

In Figures 2, 4, and 6, we plot adaptive meshes and numerical displacements by using the estimators obtained from the explicit recovery (3.3), the L^2 -projection recovery (3.2), and the $H(\text{div})$ recovery (6.1), respectively. Here the stabilization parameter is chosen as $\delta_K = h_K$ on each element $K \in \mathcal{T}_h$. Note that the constant C in the stabilization parameter γ_K in $H(\text{div})$ recovery (6.1) is taken as $C = 1$ throughout numerical experiments.

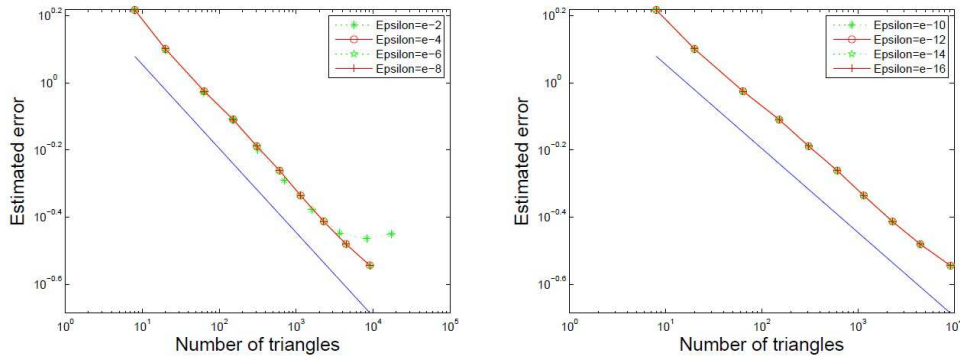


Figure 3: Estimated error of the flux against the number of elements in adaptively refined meshes for ε from 10^{-2} to 10^{-8} (left) and from 10^{-10} to 10^{-16} (right) by using the estimator from (3.3).

It is observed that strong mesh refinements occur along $x = 1$ and $y = 1$, where the estimators correctly capture boundary layers and resolve them in convection-dominated regimes. Figures 3, 5, and 7, which are respectively in correspondence to (4.1), (4.2), and (6.5), report the estimated error against the number of elements in adaptively refined meshes obtained by using estimators from flux recoveries (3.3), (3.2), and (6.1), respectively. Here $\delta_K = 16h_K$, the errors are measured in $||| \cdot |||_*$, and ε is from 10^{-2} to 10^{-16} . It is observed that the estimated errors depend on DOF uniformly in ε . The estimators work well even if Péclet number is large, and the estimated errors of all three cases are convergent. As indicated in Remark 2.2, we substitute $|||u - u_h|||_*$ with $||u - u_h||_{SUPG}$ or $|||u - u_h|||_\varepsilon$ to compute the effectivity indices. We point out that the performance of the true error $|||u - u_h|||_*$ is between that of $|||u - u_h|||_\varepsilon$ and $||u - u_h||_{SUPG}$ up to a multiple of a constant independent of h and ε . To confirm this assertion, Figure 8 illustrates $||u - u_h||_{SUPG}$, the estimated error, and $|||u - u_h|||_\varepsilon$. It is observed that, in the convection-dominated regime, the behavior of the true error is very similar to that of $|||u - u_h|||_\varepsilon$ and $||u - u_h||_{SUPG}$. Thus, it is reasonable to use $|||u - u_h|||_\varepsilon$ or $||u - u_h||_{SUPG}$ to approximate the true error $|||u - u_h|||_*$ when convection dominates. In Table 1, we show numerical results for implicit L^2 -projection recovering for $\varepsilon = 10^{-6}$, $\theta = 0.5$, and $\delta_K = 4h_K$. The effectivity indices (ratio of estimated and exact errors) are close to 1 after 8 iterations. Moreover, the estimators are robust with respect to ε .

We have checked the cases for δ_K from $\delta_K = h_K$ to $\delta_K = 16h_K$, and found that the choice of δ_K has a slight influence to the quality of the mesh. This observation indicates that adaptivity and stabilization for convection-diffusion equation is worthy of further study. In fact, the current state-of-the-art in stabilization is not completely satisfactory. In particular, the choice of stabilization parameters is still a subtle issue that is not fully understood. This is reflected either by remaining unphysical oscillations in the numerical solution or by smearing solution features too much. For more discussion on this subject, we refer to [14].

7.2 Example 2: interior and boundary layer

This model problem is one of the examples solved by Verfürth in ALF software. Let $\Omega = (-1, 1)^2$. We set the velocity field $\mathbf{a} = (2, 1)$, the reaction coefficient $b = 0$, and the source term $f = 0$ in (1.1), and consider cases for ε from 10^{-3} to 10^{-15} . The following Dirichlet boundary conditions are applied: $u(x, y) = 0$ along $x = -1$ and $y = 1$, and $u(x, y) = 100$ along $x = 1$ and $y = -1$. The exact solution of this problem is not available, which however exhibits an exponential boundary layer along the boundary $\{(x, y) : x = 1, y > 0\}$, and a parabolic interior layer along the line segment connecting points $(-1, -1)$ and $(1, 0)$. Note that the interior layer extends in the direction of the convection coefficient.

We choose the same initial mesh as in Example 1. From Figures 9, 11, and 13, which are respectively depicted by using the estimators obtained from the explicit recovery (3.3), the L^2 -projection recovery (3.2), and the $H(\text{div})$ recovery (6.1), and by choosing the stabilization parameters as $\delta_K = h_K$. It is observed that the meshes are refined in both the exponential and the parabolic layer regions, but the refinement first occurs in the region near $\{(x, y) : x = 1, y > 0\}$. The reason is that the exponential layer is much stronger than the parabolic layer. It is also observed that each one of three estimators capture the behavior of the solution pretty well, even when the singular perturbation parameter ε is very small.

Figures 10, 12, and 14 are depicted by using the estimators obtained from the flux recovery (3.3), (3.2), and (6.1), respectively, and by choosing the stabilization parameters as $\delta_K = 16h_K$. The estimated error against the number of elements in adaptively refined mesh for ε from 10^{-3} to 10^{-15} are reported. It is observed that all three estimated errors

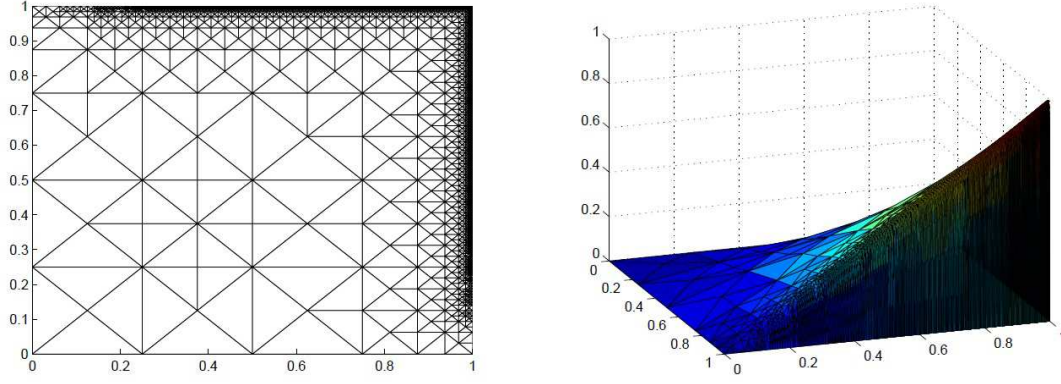


Figure 4: An adaptive mesh with 30869 triangles (left) and the approximation of displacement (piecewise linear element) on the corresponding mesh (right) for $\varepsilon = 10^{-16}$ by using the estimator from (3.2).

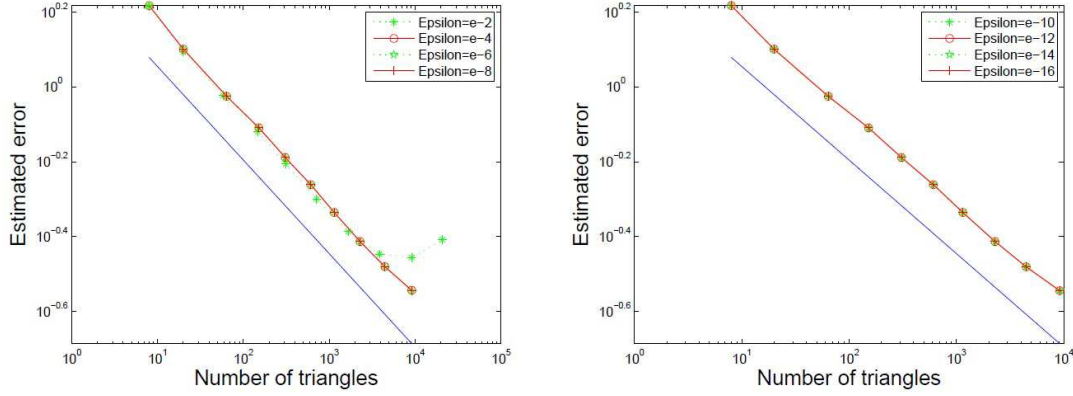


Figure 5: Estimated error of the flux against the number of elements in adaptively refined meshes for ε from 10^{-2} to 10^{-8} (left) and from 10^{-10} to 10^{-16} (right) by using the estimator from (3.2).

from respective estimators in norm $||| \cdot |||_*$ reduce uniformly in sufficiently small ε in absence of reaction term. In addition, the same convergence rates as in Example 1 are obtained. It is also noticed that the performance of the three estimators are similar.

In Table 2, data for different ε s are provided. The adaptive iterations refine elements till the layer is resolved or the TOL is met. One may observe that the performance of the error estimators depends on the TOL; the minimum mesh sizes h_{min} are of order $O(\varepsilon h_{max})$ or $O(\varepsilon)$, since the maximum mesh sizes $h_{max}(\varepsilon)$ and the initial mesh size h_0 are of similar sizes; the DOF required for resolving layers will increase when TOL and/or ε decrease; and the proposed error estimators are robust with respect to ε . On the other hand, due to the current state-of-the-art in stabilization, spurious oscillations may occur on very fine mesh, which will hence affect the quality of mesh refinement of further iterations and the rate of convergence of the method; cf. [14] and the plots for $\varepsilon = 10^{-2}$ in Figures 3 and 5.

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Table 1: Example 1: k – the number of iterations; η_k – the estimated numerical error in $||| \cdot |||_*$; err_{SUPG} and eff-index_1 – the exact error in $|| \cdot ||_{\text{SUPG}}$ and the corresponding effectivity index; and err_{APP} and eff-index_2 – the exact approximation error in $||| \cdot |||_\varepsilon$ and the corresponding effectivity index. Here $\varepsilon = 10^{-6}$, $\theta = 0.5$, and $\delta_K = 4h_K$.

k	1	2	3	4	5	6	7	8
η_k	1.6476	1.2551	0.9790	0.7580	0.6328	0.5438	0.4908	0.4573
err_{SUPG}	1.8419	1.4871	1.1914	0.9387	0.7717	0.6445	0.5492	0.4765
eff-index_1	0.8945	0.8440	0.8217	0.8075	0.8200	0.8437	0.8936	0.9598
err_{APP}	0.8204	0.6811	0.5760	0.4971	0.4519	0.4253	0.4018	0.3852
eff-index_2	2.0083	1.8427	1.6998	1.5230	1.4002	1.2786	1.2216	1.1872

Table 2: Example 2: Numerical results by (3.3) with $\delta_K = 16h_K$ and $\theta = 0.3$. In the table, ε is the singular perturbation parameter, η_k is the estimated numerical error, TOL is the given tolerance, DOF is the degrees of freedom, $h_{\max}(\varepsilon)$ and $h_{\min}(\varepsilon)$ are respectively the largest and smallest mesh sizes, and k is the number of iterations.

ε	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}
η_k/TOL	24.159	29.5353	51.0934	86.1609	154.3741
DOF	28194	15696	1999	368	75
$h_{\max}(\varepsilon)$	0.55902	0.55902	0.55902	0.55902	1.118034
$h_{\min}(\varepsilon)$	7.63e-06	3.05e-05	4.88e-04	3.91e-03	0.031250
k	20	18	14	11	8

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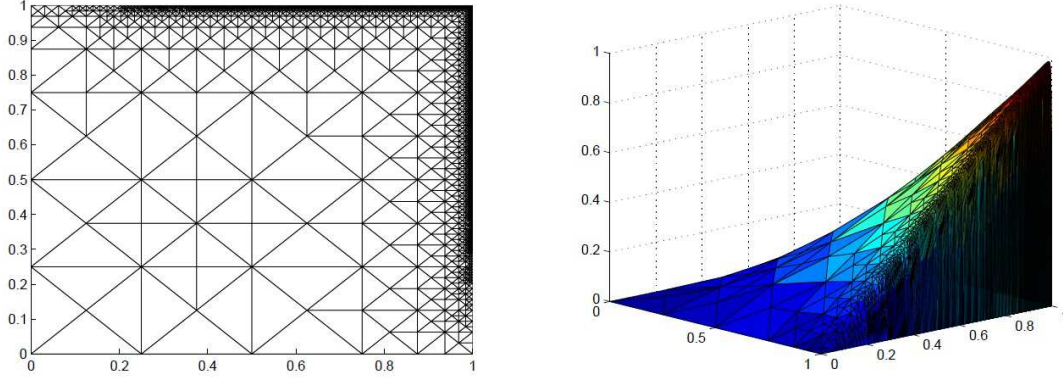


Figure 6: An adaptive mesh with 17382 triangles (left) and the approximation of displacement (piecewise linear element) on the corresponding mesh (right) for $\varepsilon = 10^{-16}$ by using the estimator from (6.1).

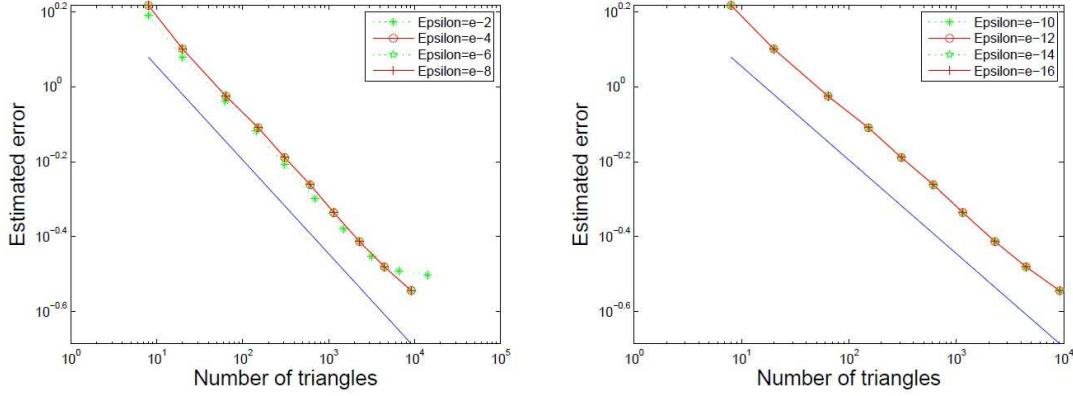


Figure 7: Estimated error of the flux against the number of elements in adaptively refined meshes for ε from 10^{-2} to 10^{-8} (left) and from 10^{-10} to 10^{-16} (right) by using the estimator from (6.1).

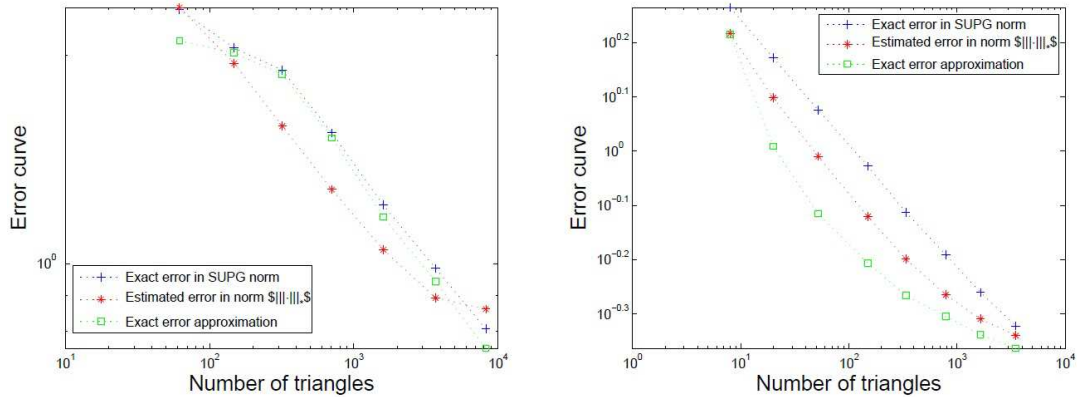


Figure 8: Exact error in SUPG norm, estimated error in norm $||| \cdot |||_*$, and exact error approximation in norm $||| \cdot |||_\varepsilon$ for explicit recovering for Example 1 with $\theta = 0.5$, $\varepsilon = 10^{-2}$, and $\delta_K = 16h_K$ (left), and $\theta = 0.5$, $\varepsilon = 10^{-6}$, and $\delta_K = 4h_K$ (right).

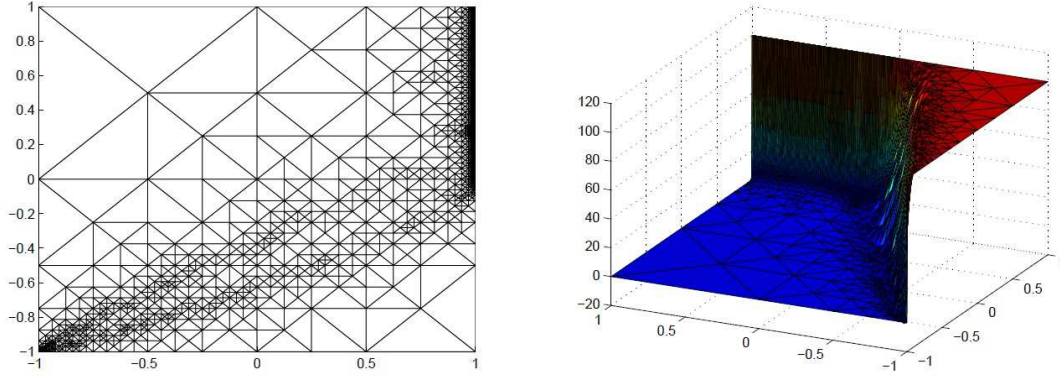


Figure 9: An adaptive mesh with 14315 triangles (left) and the approximation of displacement (piecewise linear element) on the corresponding mesh (right) for $\varepsilon = 10^{-11}$ by using the estimator from (3.3).

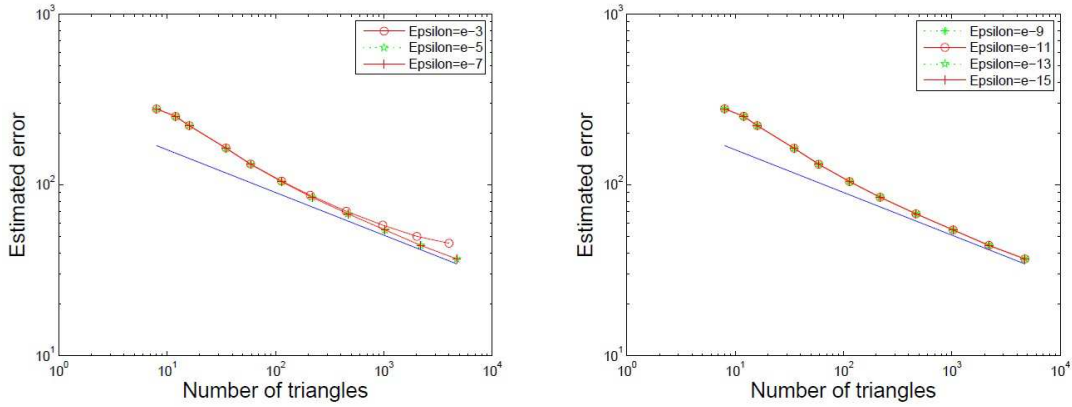


Figure 10: Estimated error of the flux against the number of elements in adaptively refined meshes for ε from 10^{-3} to 10^{-7} (left) and from 10^{-9} to 10^{-15} (right) by using the estimator from (3.3).

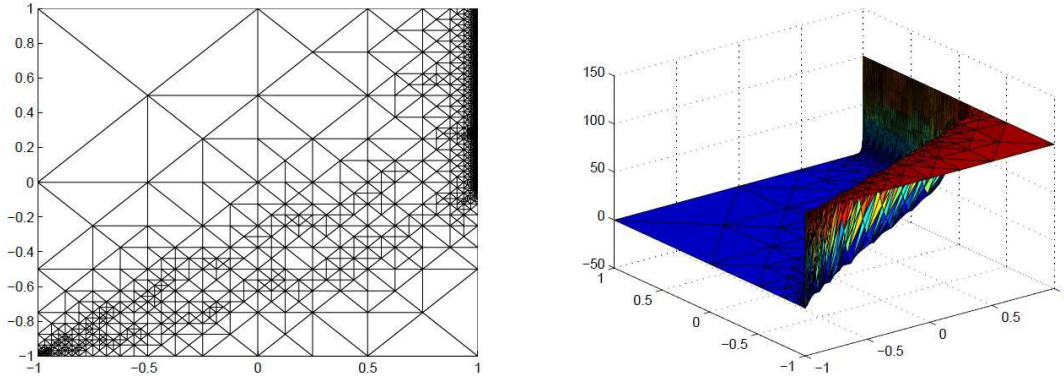


Figure 11: An adaptive mesh with 7761 triangles (left) and the approximation of displacement (piecewise linear element) on the corresponding mesh (right) for $\varepsilon = 10^{-11}$ by using the estimator from (3.2).

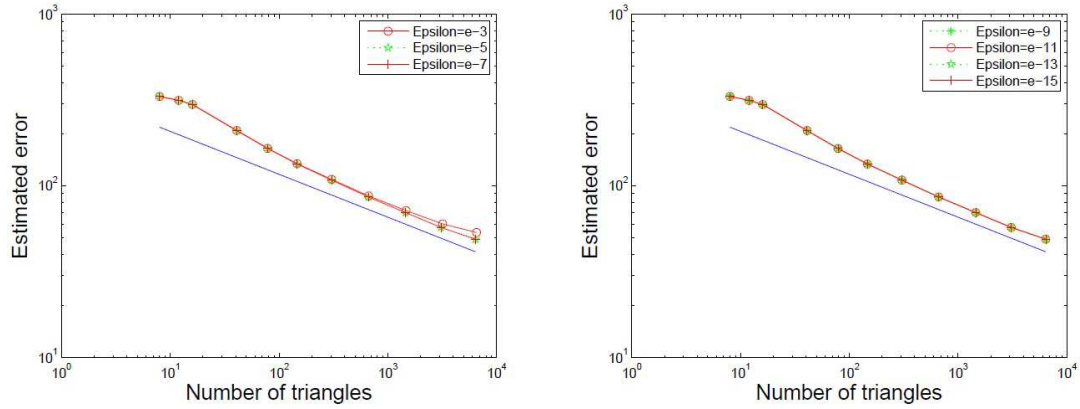


Figure 12: Estimated error of the flux against the number of elements in adaptively refined meshes for ε from 10^{-3} to 10^{-7} (left) and from 10^{-9} to 10^{-15} (right) by using the estimator from (3.2).

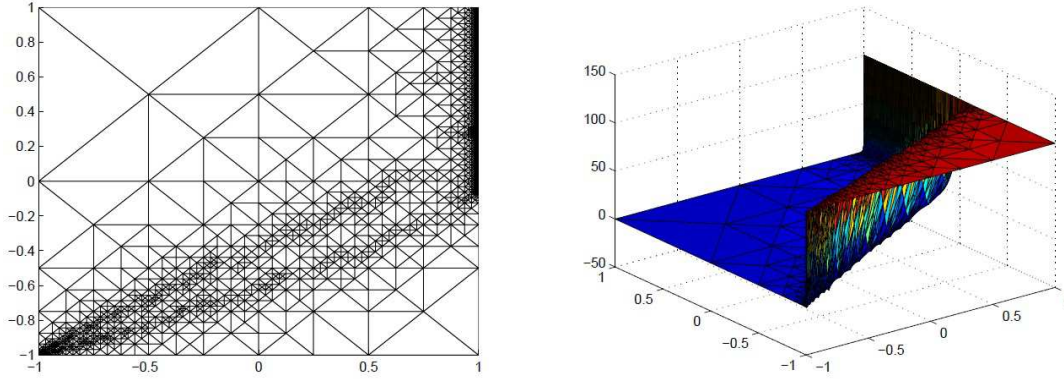


Figure 13: An adaptive mesh with 27309 triangles (left) and the approximation of displacement (piecewise linear element) on the corresponding mesh (right) for $\varepsilon = 10^{-11}$ by using the estimator from (6.1).

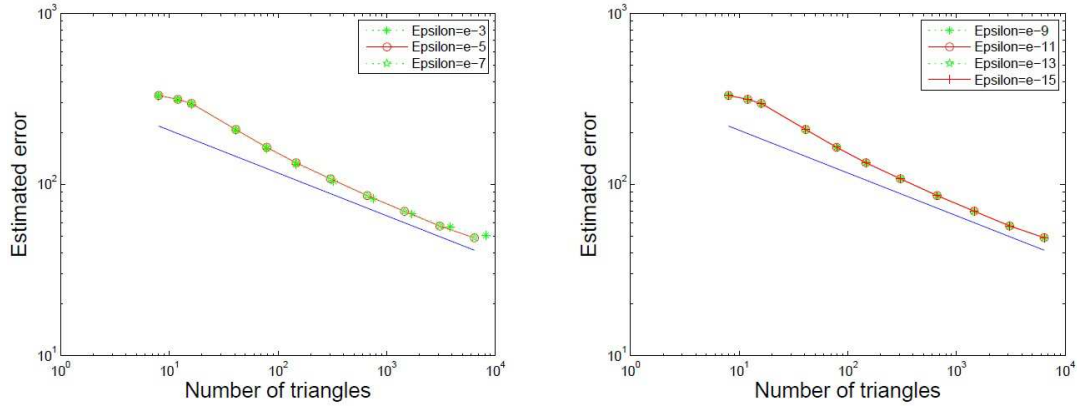


Figure 14: Estimated error of the flux against the number of elements in adaptively refined meshes for ε from 10^{-3} to 10^{-7} (left) and from 10^{-9} to 10^{-15} (right) by using the estimator from (6.1).